*J. Fluid Mech.* (1973), *vol.* 60, *part* 4, *pp.* 625–639
*Printed in Great Britain*


# The bursting of two-dimensional drops in slow viscous flow

By J. D. BUCKMASTER
Mason Laboratory, Yale University, New Haven, Connecticut
AND J. E. FLAHERTY
Mathematics Department, Rensselaer Polytechnic Institute, Troy, N.Y.

(Received 7 November 1972)

We consider the deformation of two-dimensional drops when immersed in a slow viscous corner flow. The problem is formulated as one of analytic function theory and simplified by assuming that both the drop and the exterior fluid have the same viscosity. An approximate analysis is carried out, in which the conditions at the interface are satisfied in an average sense, and this reveals the following features of the solution. A drop of given physical properties (volume, surface tension and viscosity), when immersed in a corner flow, has no steady equilibrium shape if the rate of strain of the applied flow is too large. On the other hand, if the rate of strain is small enough for a steady solution to exist, then in general there are *two* possible solutions. These features are confirmed by formulating the exact problem in terms of a nonlinear integro-differential equation, which is solved numerically.

## 1. Introduction

In 1934 Taylor described some experiments in which drops of viscous liquid were subjected to a viscous corner flow and a viscous shear flow, at low Reynolds numbers. Similar experiments were reported by Rumscheidt & Mason (1961), and a wide range of interesting phenomena were described, including the existence of steady pointed drops, bursting pointed drops and bursting rounded drops.

In the experiments a single drop was placed in a viscous bath, and then the effects of increasing flow velocity, on both the shape and behaviour of the drop, were observed. This procedure was carried out for a variety of fluids, and for the corner flow, the observations can be roughly summed up as follows. At small straining rates, all the drops are essentially spheroidal in shape, being elongated in the direction of the flow; this elongation increases with increasing rate of strain. For drops that are relatively inviscid in comparison with the surrounding fluid there is a sudden qualitative change in shape at a critical straining rate, in that the ends abruptly become pointed. Further increases in strain merely increase the elongation, and these pointed configurations survive up to the maximum rates of strain attainable in the apparatus. At higher drop viscosities, comparable with that of the exterior fluid, steady pointed shapes are also observed at sufficiently large straining rates. However, there is a second critical rate of strain at which the drops 'burst'; that is, they rapidly increase in length with passing time (unsteady flow), at the same time remaining pointed. When the drop viscosity is very large, pointed shapes are never seen. Bursting occurs at sufficiently high rates of strain, but both before and after bursting the ends are always rounded. Bursting and points are also observed in the shear flow.

The bursting phenomenon is of fundamental interest, lying at the heart, in fact, of Taylor's motivation for his work, namely emulsion formation. Several plausible explanations may be conjectured. It could, for example, be a stability question; that is, at sufficient straining rates it is conceivable that the steady shapes become unstable to disturbances that must inevitably exist in a real flow. Implicit in this view is the idea that a mathematical solution of the problem *does* exist for some range of straining rates greater than the bursting value, but this solution is never realized in practice. A second possibility is that the phenomenon is simply one of non-existence – that is, there is an upper bound on the straining rates for which a steady solution exists. A third possibility is that the solution has several unconnected branches, the branch that one is on depending on the manner in which the experiment is set up. Bursting could then be associated with an upper bound for the current branch. Naturally we do not rule out the chance that the mathematical description involves a mixture of these three possibilities. For example, non-existence may occur beyond some limiting straining rate, but before this limit is reached instability may destroy the solution.

In order to decide conclusively which, if any, of these possibilities is responsible for bursting, it is necessary to model, exactly, the flows set up by Taylor, and discuss fully the solutions and their stability. This is a formidable problem however, the overwhelming difficulty arising from the fact that these flows are genuinely three-dimensional. In the face of this, two simplified models naturally suggest themselves: plane flows and axisymmetric flows. It is conceivable that an understanding of these will lead to an understanding of the real situation. The work of Taylor (1964*a*) and Buckmaster (1972, 1973) exploits the simplifications of axisymmetric flow (made even simpler by the assumption of a slender drop), whereas Richardson (1968) considers plane drops. Plane flows have the advantage that the powerful techniques of analytic function theory may be used, but there is always the danger that the solutions have features uncharacteristic of real flows. In this connexion it is worth while comparing the qualitative predictions of plane flows with those of axisymmetric flow (where possible), since for shear flows we have no choice but to model them with the former.

There have been several theoretical attacks on problems of this kind, but only the references already cited deal with large deformations from circular or spherical shapes, and this appears to be an essential ingredient in generating points or bursting. Richardson (1968) examined plane inviscid drops in both corner flows and shear flows, and was able to generate exact solutions. None of these solutions exhibit points, nor do they provide any evidence of bursting. The latter is in agreement with experiment and, indeed, the present authors are of the opinion that bursting can only occur, when the applied flow is linear, if the drop viscosity is finite. The absence of points, on the other hand, is more likely to be a creature of the plane-flow model. Thus, equilibrium for a plane drop with two pointed ends is unlikely, since as it thins out the interface curvature *decreases*. Intuitively, we might expect that it is an increasing stress associated with *increasing* curvature that can provide the resistance to deformation associated with increasing straining rates. Moreover, as Richardson has pointed out, a discontinuity in the interface slope of a plane drop applies a point force to the surrounding fluid by virtue of interfacial tension. This force completely dominates the flow in the neighbourhood of the corner and is quite atypical of the pointed drops generated by Taylor, for which the discontinuities are three-dimensional points, and not edges.

Buckmaster (1972, 1973) examined axisymmetric drops in a corner flow, and the second of these studies is concerned with viscous drops. By assuming that the surface tension and drop viscosity are small, so that the drop is slender, a rational analytic description is possible. A central result is that there is no steady solution of the Stokes-flow equations if

$$K \equiv T/aC(\mu_o \mu_i)^{\frac{1}{2}} < 8,$$

where $T$ is surface tension, $\mu_o$ is the viscosity of the outer fluid, $\mu_i$ is the drop viscosity, $C$ is the velocity gradient that characterizes the corner flow and $2a$ is the length of the drop. The implications of this for the experiments carried out by Taylor are suggestive. For a given pair of fluids (i.e. fixed $T$, $\mu_o$ and $\mu_i$) the drop is increasingly deformed by increasing $C$. An increase in $a$ characterizes this deformation, so that, provided $\mu_i$ is not very small, $K$ will eventually become smaller than 8, and a steady state will cease to be viable. It is natural to identify this with the onset of bursting, particularly since this criterion predicts that inviscid drops will never burst. Of course, a steady analysis can give no clues as to the nature of the subsequent unsteady flow. Confidence that we have the right criterion for bursting would be increased if it could be shown that, when $K < 8$, there is an *unsteady* solution corresponding to a perpetually elongating drop. Such a solution is described by Buckmaster (1973).

Several criticisms may be justifiably levelled at the slender-drop analysis. From a mathematical point of view, the asymptotic analysis is entirely formal, and there are acknowledged difficulties near the ends, for which a resolution is only suggested. More significantly, it is not possible, with such an analysis, to follow the deformation from the ball shape at $C = 0$ to the point where a steady solution does not exist. It is conceivable that the slender-drop solutions correspond to a physically unattainable branch.

In the present paper we examine plane viscous drops in a corner flow, primarily by numerical means. We are concerned with following the deformation of the drop from the circle at zero straining rates to at least the point of non-existence (assuming it exists). Furthermore, we want to continue the response curve through this critical point, if possible, in order to establish non-uniqueness (i.e. two possible configurations for a given straining flow). To the author's knowledge, non-uniqueness in steady slow viscous flow has only once been reported before (Buckmaster 1973), so that additional evidence would be welcome. It would provide a counter example to any speculation that the uniqueness results of

Keller, Rubenfeld & Molyneux (1967) can be extended to free-surface situations, for example.

There are several reasons why we examine a plane drop. The primary one is that it leads to a relatively simple problem. But, in addition, we feel that both the plane and axisymmetric problems are worth systematically exploring, especially if physical reasons can be given for any qualitative features that are revealed.

It is worth remarking that the phenomena of pointed interfaces and bursting can occur in other situations. For example, an uncharged conducting drop placed in an electric field abruptly forms pointed ends at a critical field strength (Taylor 1964*b*). In addition, it is probable that skirt formation on spherical gas caps is a bursting phenomenon (Wegener, Sundell & Parlange 1971). Note that the formation of skirts is essentially a two-dimensional problem, raising the possibility that there may be direct applications of plane flow analyses such as the present one.

We start the description in § 2 by formulating the general problem of a viscous drop placed in a corner flow, as one of analytic function theory. Substantial simplification is achieved by assuming that the drop and the outer liquid both have the same viscosity. There is no reason to believe that this special case is atypical. In § 3 an approximate solution is derived. The drop is assumed to have an elliptic shape and the eccentricity is determined by satisfying the interfacial conditions in some average sense. In § 4 the problem is reduced to a single non-linear integro-differential equation for the drop shape, which is solved numerically. Sections 3 and 4 are mutually self-supporting, showing that an approximate analysis of this kind can be very accurate and providing evidence that the numerical results are correct. Finally, in § 5, the results are summarized and explained.

## 2. Formulation as a problem in analytic function theory

Consider the situation described by figure 1 in which a drop with viscosity $\mu_i$ is immersed in a corner flow of a liquid with viscosity $\mu_0$. The Reynolds number $\rho Ca^2/\mu$ is assumed to be small enough to justify the neglect of inertia terms, so that the corner flow can be described by

$$u = Cx, \quad v = -Cy, \quad p = p_\infty \tag{2.1}$$

and the presence of the drop disturbs this. The undisturbed flow is *not* a solution of the Navier–Stokes equations, so that our description will not be uniformly valid (it breaks down at infinity), but this does not interfere with our basic aim, which is to calculate the shape of the drop.

Since the stream function satisfies the biharmonic equation, it can be represented in the usual way as

$$\psi(x,y) = \mathrm{Re}\,[\bar{z}\phi(z)+\chi(z)], \tag{2.2}$$

so that

$$-v+iu = \phi(z)+z\overline{\phi'(z)}+\overline{\chi'(z)}, \tag{2.3}$$

where $\phi(z)$ and $\chi(z)$ are analytic functions of the complex variable $z = x+iy$. Richardson (1968) used this approach in his study of inviscid drops ($\mu_i = 0$),

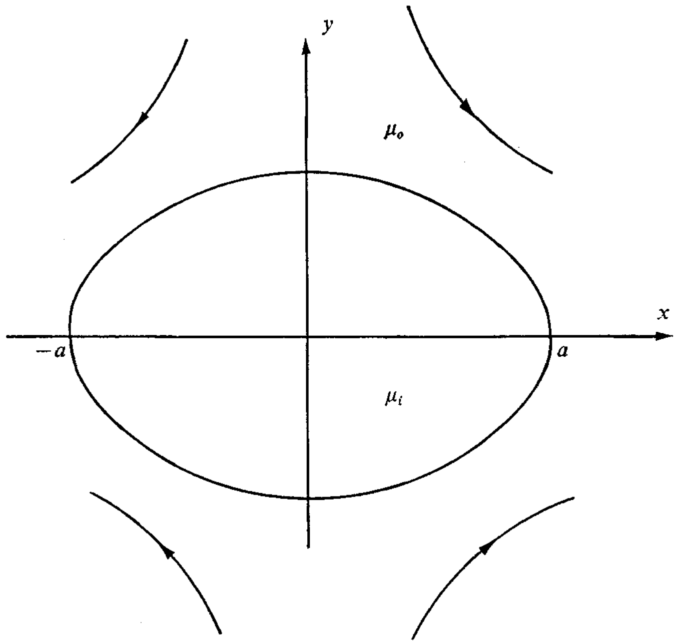


FIGURE 1. Drop in corner flow.

and in particular derived the appropriate interfacial conditions for such drops. We shall follow his analysis with appropriate modifications to allow for the effects of non-zero $\mu_i$. Thus, referring to that paper, the stress balance on the interface is

$$T\frac{d}{ds}\left(\frac{dz}{ds}\right) = \delta\left(2\mu\frac{d}{ds}(\phi - z\overline{\phi'} - \overline{\chi'})\right),$$

where $T$ is the surface tension and $\delta$ refers to the jump across the interface. Upon integration (absorbing the constant of integration into $\phi_i$), this yields

$$\tfrac{1}{2}T(dz/ds) = \mu_o(\phi_o - z\overline{\phi'_o} - \overline{\chi'_o}) - \mu_i(\phi_i - z\overline{\phi'_i} - \overline{\chi'_i}) \quad \text{on } C. \tag{2.4}$$

In addition, the velocity is continuous on the interface, so that

$$\phi_o + z\overline{\phi'_o} + \overline{\chi'_o} = \phi_i + z\overline{\phi'_i} + \overline{\chi'_i} \quad \text{on } C. \tag{2.5}$$

Equations (2.4) and (2.5) are correct whether the interface is stationary or not, but for a steady situation they are supplemented by the condition

$$\psi = \mathrm{Re}\,(\bar{z}\phi + \chi) = \text{constant} \quad \text{on } C. \tag{2.6}$$

These equations, together with appropriate conditions on $\phi$ and $\chi$ at infinity are sufficient, in principle, to determine $\phi, \chi$ and the shape of the interface $C$. In practice however, they present a formidable problem, so that we shall introduce a major simplification, namely

$$\mu_i = \mu_o \quad (= \mu). \tag{2.7}$$

It does not seem likely that this is atypical of the general situation, so that any qualitative features that we determine are of interest. The simplification arises because only the normal stress is then discontinuous across the interface.

Equations (2.4) and (2.5) now yield

$$\phi_o - \phi_i = \frac{T}{4\mu}\frac{dz}{ds} \quad \text{on } C. \tag{2.8}$$

Moreover, $$\chi_o' - \chi_i' = -\frac{T}{4\mu}\frac{d\bar{z}}{ds} - \frac{T}{4\mu}\bar{z}\frac{ds}{dz}\frac{d}{ds}\left(\frac{dz}{ds}\right),$$

which on multiplication by $dz/ds$ and integration (absorbing the constant of integration into $\chi_i$) yields

$$\chi_o - \chi_i = -\frac{T}{4\mu}\bar{z}\frac{dz}{ds} \quad \text{on } C. \tag{2.9}$$

Equations (2.8) and (2.9) are completely equivalent to (2.4) and (2.5) and have the advantage that, if the shape of the drop is known, explicit solutions for $\phi$ and $\chi$ can immediately be written down. Note that we have not yet used the condition that for a steady flow the drop interface is a streamline (equation (2.6)). Thus (2.8) and (2.9) are valid even if the interface is moving, and could be used, for example, to calculate the motion of a drop of initially known shape.

The required behaviour of $\phi$ and $\chi$ for large $z$, consistent with (2.1), is

$$\phi(z) \sim -ip_\infty z/4\mu, \quad \chi(z) \sim -\tfrac{1}{2}iCz^2 \quad \text{as} \quad |z| \to \infty,$$

so that from (2.8) and (2.9) we have, by virtue of the Plemelj formula,

$$\left.\begin{aligned} \phi(z) &= -\frac{T}{4\mu}\frac{1}{2\pi i}\int_C \frac{(dt/ds)\,dt}{t-z} - \frac{ip_\infty}{4\mu}z, \\ \chi(z) &= \frac{T}{4\mu}\frac{1}{2\pi i}\int_C \bar{t}\frac{(dt/ds)\,dt}{t-z} - \frac{iC}{2}z^2 + ik, \end{aligned}\right\} \tag{2.10}$$

where $k$ is a real constant.† Thus the problem is to find a drop shape such that when $\phi$ and $\chi$ are calculated from (2.10) they satisfy the condition (2.6) on the interface. (The constant in (2.6) is chosen to be zero.) In the next section an approximate solution of this problem is obtained.

## 3. Approximate solution

An approximate solution of the problem formulated in §2 can be derived by supposing that the drop is one of a one-parameter family of shapes, the value of the parameter to be determined by satisfying the conditions on the boundary in some average sense. An approach of this kind was very successfully taken by Taylor (1964*b*) in describing the equilibrium configuration of conducting drops in a uniform electric field. He assumed that the drop was a prolate spheroid and then determined the internal pressure and the eccentricity by (in one calculation) satisfying the stress condition at the ends and the equator. Numerical calculations by Brazier-Smith (1971) show that this gives remarkably accurate results. In the present problem we take advantage of the fact that the behaviour of an analytic function far from a finite body is related to the average of the function

† Like Richardson (1968), for symmetry reasons we omit logarithmic terms.

over the boundary. Specifically, Cauchy's integral formula can be used to relate the coefficients in the Laurent expansion to certain weighted integrals of the function over the boundary. Consequently, if we wish to ensure that two different analytic functions are equal, in some average sense, on the boundary, it is appropriate to equate as many of the leading coefficients in their Laurent expansions as possible.

Thus, suppose that the drop can be represented approximately by an ellipse. The exterior of the ellipse is related to the exterior of the unit circle in the $\zeta$ plane by the mapping

$$z = w(\zeta) = A(\zeta + B/\zeta). \tag{3.1}$$

For the remainder of this section we shall only be concerned with $\phi_o$ and $\chi_o$ so the subscript will be dropped. Writing

$$\Phi(\zeta) = \phi(w(\zeta)), \quad \mathrm{X}(\zeta) = \chi(w(\zeta)),$$

the tangency condition (2.6) is (with the constant equal to zero)

$$A(1/\zeta + B\zeta)\,\Phi(\zeta) + \mathrm{X}(\zeta) = -A(\zeta + B/\zeta)\,\overline{\Phi(1/\bar{\zeta})} - \overline{\mathrm{X}(1/\bar{\zeta})} \quad \text{on} \quad |\zeta| = 1. \tag{3.2}$$

The left-hand side of this equation is analytic in $|\zeta| \geqslant 1$, whereas the right-hand side is meromorphic in $|\zeta| \leqslant 1$, with a pole at the origin. Consequently, the right-hand side is the analytic continuation of the left into $|\zeta| < 1$, so that (3.2) is, in fact, valid in $|\zeta| \leqslant 1$. Writing $\zeta = 1/\bar{\eta}$ and taking the complex conjugate yields

$$A(\eta + B/\eta)\,\overline{\Phi(1/\bar{\eta})} + \overline{\mathrm{X}(1/\bar{\eta})} = -A(1/\eta + B\eta)\,\Phi(\eta) + \mathrm{X}(\eta) \quad \text{in} \quad |\eta| \geqslant 1$$

and comparison with (3.2) then shows that the latter is valid everywhere.

Now as $|\zeta| \to \infty$

$$\Phi(\zeta) \sim -i\frac{p_\infty A\zeta}{4\mu} + \frac{\Phi_{-1}}{\zeta} + O(1/\zeta^3)$$

($\Phi_{-1}$ imaginary) and we choose the arbitrary constant $k$ in (2.10) so that

$$\mathrm{X}(\zeta) \sim -\tfrac{1}{2}iCA^2\zeta^2 + O(1/\zeta^2).$$

It follows that, as $\zeta \to 0$,

$$\overline{\Phi(1/\bar{\zeta})} \sim \frac{ip_\infty}{4\mu}\frac{A}{\zeta} + O(\zeta), \quad \overline{\mathrm{X}(1/\bar{\zeta})} \sim \frac{iC}{2}\frac{A^2}{\zeta^2} + O(\zeta^2),$$

and this determines the nature of the pole of the right-hand side of (3.2). Since the function represented by both the left and right is analytic outside the unit circle with known behaviour at infinity, and meromorphic in the interior with a known pole at the origin, it follows that

$$A\left(\frac{1}{\zeta} + B\zeta\right)\Phi(\zeta) + \mathrm{X}(\zeta) = \left(AB\Phi_{-1} - \frac{ip_\infty}{4\mu}A^2\right) - \left(\zeta^2 + \frac{1}{\zeta^2}\right)\left(\frac{ip_\infty}{4\mu}A^2B + \frac{iC}{2}A^2\right). \tag{3.3}$$

Thus, to satisfy the tangency condition, $\Phi$ and $\mathrm{X}$ are related to each other, and in particular

$$A\Phi_{-1} + AB\Phi_{-3} + \mathrm{X}_{-2} = -\left[\frac{ip_\infty}{4\mu}A^2B + \frac{iC}{2}A^2\right], \tag{3.4}$$

where $\Phi_{-n}$ is the coefficient of $\zeta^{-n}$ in the expansion of $\Phi$ for large $|\zeta|$ (similarly for $X_{-n}$).

In addition, $\Phi$ and X are determined in terms of the parameters $A$ and $B$ by the expressions (2.10). Thus $\Phi_{-n}$, and $X_{-n}$ are easily determined by expanding $[t-w(\zeta)]^{-1}$ for large $|\zeta|$, whence

$$\left.\begin{aligned} \Phi_{-1} &= \frac{T}{4\mu}\frac{1}{2\pi i}\frac{1}{A}\int_C dt\frac{dt}{ds}-\frac{ip_\infty}{4\mu}AB, \\ \Phi_{-3} &= \frac{T}{4\mu}\frac{1}{2\pi i}\left[\frac{-B}{A}\int_C dt\frac{dt}{ds}+\frac{1}{A^3}\int_C dt\,t^2\frac{dt}{ds}\right], \\ X_{-2} &= -\frac{T}{4\mu}\frac{1}{2\pi i}\frac{1}{A^2}\int_C dt\,t\bar{t}\frac{dt}{ds}-\frac{iC}{2}A^2B^2. \end{aligned}\right\} \tag{3.5}$$

All of these integrals can be expressed as integrals round the unit circle, and then substituting into (3.4) yields

$$4\pi\mu\frac{CA}{T} = \int_0^{2\pi} d\theta\,\frac{B^3\cos 4\theta-2B^2\cos 2\theta+B}{[1+B^2-2B\cos 2\theta]^{\frac{1}{2}}}. \tag{3.6}$$

If $2a$ is the length of the major axis, $2b$ that of the minor and

$$e = 1-b/a,$$

we have

$$B = e/(2-e), \quad A = \tfrac{1}{2}a(2-e).$$

Thus (3.6) provides a relation between the straining parameter $\mu Ca/T$ and the deformation $e$, which allows the interfacial conditions to be satisfied in an average sense. This relation is plotted in figure 2.

There are two points that we should like to make about figure 2. In the first place there is a maximum value of $\mu Ca/T$ beyond which there is no solution. Since this parameter can always be made arbitrarily large in an experiment, by simply increasing $C$, it follows that there is no steady solution for a sufficiently high rate of strain (cf. the discussion of slender drops in the introduction). Second, $\mu Ca/T$ is a monotone function of $e$, and this has important implications for the numerical work of §4.

Although figure 2 is a perfectly correct way of representing our results, it does not lend itself to comparison with a realistic experiment. In an experiment we should take a drop of fixed volume (area), viscosity and surface tension and observe its shape for various values of $C$. Thus a more appropriate length scale than $a$ is one that does not vary during the experiment. If we take as this length scale the square root of $S$, the drop area, we get the result shown in figure 3 as the approximate solution (broken line). For a given value of the parameter $\mu C\sqrt{S}/T$, there is not a unique solution for the eccentricity. Rather, for straining rates sufficiently small for existence, there are two possible solutions. This non-uniqueness, which is intimately related to the question of non-existence, has its counterpart in the slender-drop analysis of Buckmaster (1973).

The procedure for improving the approximate solution described here is, in principle, straightforward. Thus a better approximation could be obtained by assuming that the drop can be mapped on to the unit circle by the function

$$w(\zeta) = A(\zeta+B/\zeta+D/\zeta^3),$$

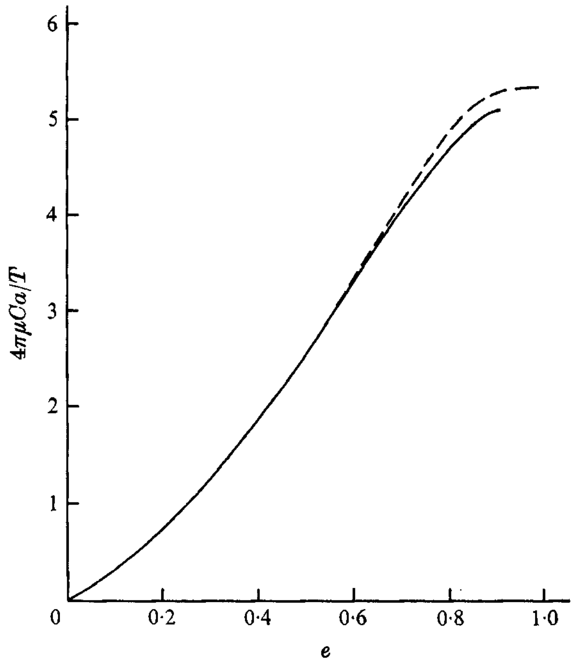


FIGURE 2. Response diagram. ——, numerical; – – –, approximate.

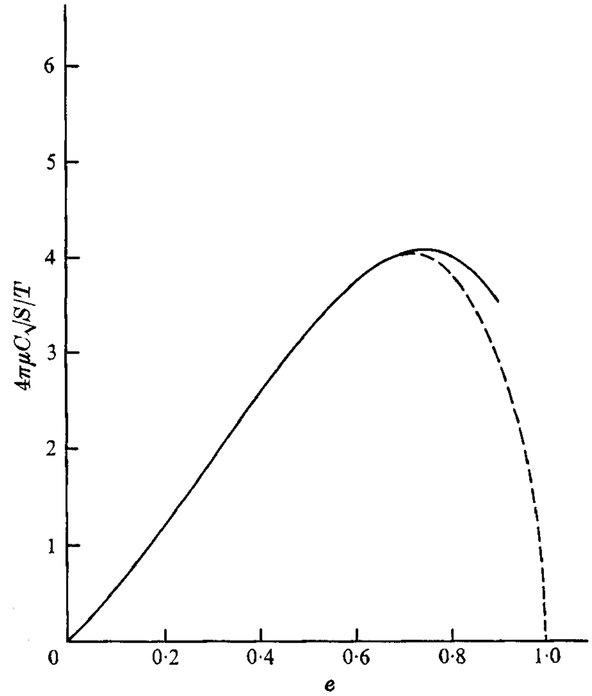


FIGURE 3. Response diagram. ——, numerical; – – –, approximate.

where now there are two parameters to be determined, $B$ and $D$. The extra degree of freedom enables us to equate more terms in the Laurent expansions. However, we do not pursue this line. It is of great interest to determine whether or not the non-uniqueness is genuine, or merely a creature of the approximate analysis, so that in the next section we describe an exact numerical solution of our problem.

## 4. Numerical solution

In this section, the problem formulated in §2 is reduced to a single integro-differential equation for the drop shape. We follow this with a description of the numerical procedure required to solve it.

The derivation of the governing equation is a straightforward exercise. Suppose that the drop is described by

$$y = \pm f(x), \quad |x| \leqslant a. \tag{4.1}$$

Equations (2.10) then determine $\phi$ and $\chi$ in terms of integrals which contain $f$ and its first derivative. This enables us to write down an integral expression for the stream function $\psi$. Setting this equal to zero for points in the set

$$\{(x, \pm f(x)) \,|\, |x| \leqslant a\}$$

then yields an integro-differential equation for $f(x)$, namely

$$\begin{aligned} 4\pi\mu\frac{Ca}{T}x_0F_0 = &\int_{-1}^{+1} dx\,\frac{-2(x-x_0)^2F' + (x-x_0)(F+F_0)(1-F'^2)}{[1+F'^2]^{\frac{1}{2}}[(x-x_0)^2+(F+F_0)^2]} \\ &+\int_{-1}^{+1} dx\,\frac{2(x-x_0)^2F' + (x-x_0)(F_0-F)(1-F'^2)}{[1+F'^2]^{\frac{1}{2}}[(x-x_0)^2+(F-F_0)^2]} \quad (|x_0| \leqslant 1), \end{aligned} \tag{4.2}$$

where $F(x) = f(ax)$, $|x| \leqslant 1$ and $F_0 \equiv F(x_0)$. The problem is, given the parameter $\mu Ca/T$, to find a non-trivial function $F(x)$ which satisfies this equation. The parameter was really forced upon us since we do not know *a priori* the relation between $a$ and $S$. Fortunately, it is the most appropriate parameter for the numerical integration, since we can expect a unique dependence of the shape thereon.

The integral equation (4.2), as written, is difficult to approximate numerically because of the singularities in $F'$ at the ends $x = \pm 1$. To avoid this difficulty and at the same time build in an appropriate distribution of step sizes, we transform to elliptic polar co-ordinates

$$x(\lambda) = r(\lambda)\cos\theta(\lambda), \quad F(\lambda) = r(\lambda)\sin\theta(\lambda), \tag{4.3}$$

$$\tan\theta(\lambda) = \epsilon\tan\lambda \quad (0 \leqslant \lambda \leqslant \tfrac{1}{2}\pi).$$

Here $\epsilon$ is a parameter with value between zero and one, chosen to yield the greatest numerical accuracy. $\epsilon$ may be interpreted as the minor to major axis ratio of an ellipse. Values close to one are appropriate when the drop is only slightly different from a circle, but as it gets stretched out with increasing values of $\mu Ca/T$, smaller values of $\epsilon$ are appropriate.

With the transformation (4.3) implied, and making use of the symmetry of the problem, (4.2) can be written as

$$H(r(\lambda_0)) = 0 = 4\pi\mu\frac{Ca}{T}F_0 + 2\int_0^{\frac{1}{2}\pi} d\lambda[G(x, F, x_0, F_0, \dot{x}, \dot{F}) - G(x, F, x_0, -F_0, \dot{x}, \dot{F})], \tag{4.4}$$

where

$$G(x, F, x_0, F_0, \dot{x}, \dot{F}) = \frac{4x\dot{x}\dot{F}(F+F_0)^2 + (F+F_0)(\dot{x}^2-\dot{F}^2)[x^2-x_0^2-(F+F_0)^2]}{(\dot{x}^2+\dot{F}^2)^{\frac{1}{2}}[(x^2-x_0^2)^2+2(x^2+x_0^2)(F+F_0)^2+(F+F_0)^4]}. \tag{4.5}$$

Here a dot denotes differentiation with respect to $\lambda$ and a subscript zero implies evaluation of the function at $\lambda = \lambda_0$.

At $\lambda = \lambda_0$ the integrand in (4.4) is indeterminate (removable singularity) and is given by

$$G(x_0, F_0, x_0, F_0, \dot{x}_0, \dot{F}_0) - G(x_0, F_0, x_0, -F_0, \dot{x}_0, \dot{F}_0) = \frac{(x_0^2 - F_0^2)\dot{x}_0\dot{F}_0 - x_0F_0(\dot{x}_0^2 - \dot{F}_0^2)}{2x_0(x_0^2+F_0^2)(\dot{x}_0^2+\dot{F}_0^2)^{\frac{1}{2}}},$$

which at $\lambda_0 = \frac{1}{2}\pi$ must be written as

$$G(0, F_0, 0, F_0, \dot{x}_0, 0) - G(0, F_0, 0, -F_0, \dot{x}_0, 0) = -\frac{1}{2|\dot{x}_0|}\left(\frac{\dot{x}_0}{F_0} + \ddot{F}_0\right).$$

We view (4.4) as a nonlinear integral equation for $r(\lambda)$ in the interval $[0, \frac{1}{2}\pi]$, where $r(0) = 1$. It may be solved numerically for prescribed values of $\epsilon$ and $\mu Ca/T$ using an iterative procedure. Thus a uniform grid with spacing $\pi/2N$ is introduced, and with an initial guess for $r(j\pi/2N)$, $j = 1, 2, \ldots, N$, the function $H$ (equation (4.4)) is evaluated for $\lambda_0 = j\pi/2N$, $j = 1, \ldots, N$. Derivatives of $r$ are calculated by approximating $r$ by cubic splines, and Simpson's rule is used to evaluate the integrals. Successive iterates for $r(j\pi/2N)$ are then obtained using a linear interpolation scheme (Robinson 1966; Mancino 1967). It is important to start with a good initial guess for $r(j\pi/2N)$, and for small values of $\mu Ca/T$ this can be obtained from the approximate analysis of § 3. Once a solution has been obtained for a particular value of $\mu Ca/T$, this may be extrapolated to obtain the initial guess for a slightly larger value of the parameter, and in this way the whole range of values can be covered.

As discussed in § 3 it is appropriate to cast the results in terms of a constant-area drop. Since the area $S$ is given by

$$\frac{\sqrt{S}}{a} \equiv \alpha = 2^{\frac{1}{2}}\left[\int_0^{\frac{1}{2}\pi} r^2\, d\lambda\right]^{\frac{1}{2}},$$

the parameter

$$4\pi(\mu C\sqrt{S}/T) = \alpha 4\pi(\mu Ca/T)$$

can be calculated from the numerical solution, using Simpson's rule. In addition, in order to obtain a sequence of pictures of the deformation of a single constant-area drop, it is necessary to scale the co-ordinates of the interface $(x(\lambda), y(\lambda))$ by a multiplicative factor $1/\sqrt{S}$.

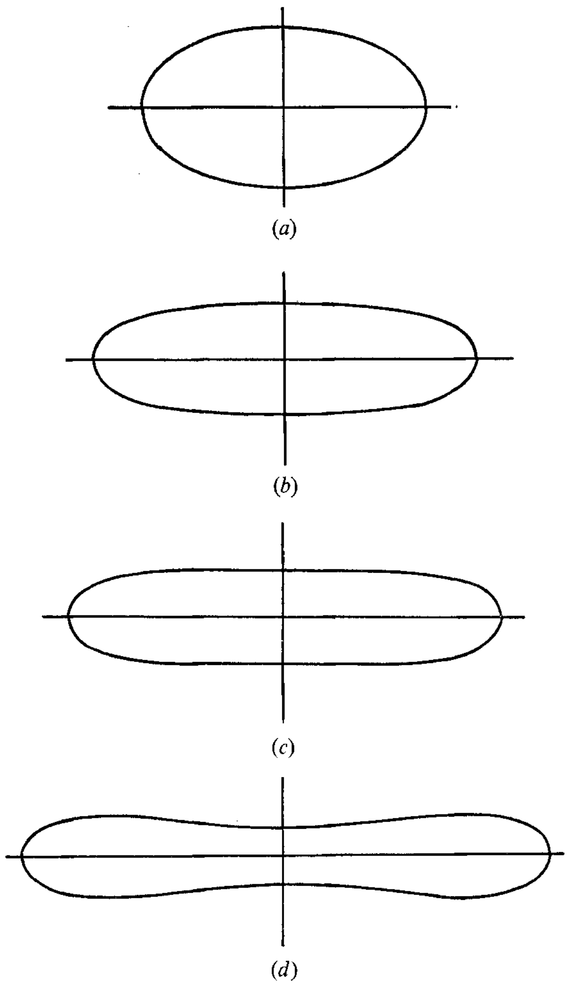


FIGURE 4. Drop shapes. (*a*) $\Omega = 2{\cdot}80$, $e = 0{\cdot}44$. (*b*) $\Omega = 4{\cdot}06$, $e = 0{\cdot}71$. (*c*) $\Omega = 4{\cdot}05$, $e = 0{\cdot}79$. (*d*) $\Omega = 3{\cdot}67$, $e = 0{\cdot}89$.

## 5. Results and concluding remarks

Figures 2 and 3, as well as showing the results of the approximate analysis, also describe the corresponding numerical results. There is substantially good agreement, except for very slender drops. It may be noted that numerical results are only shown for values of the abscissa less than about 0·9. This limit corresponds, roughly, to the most slender drop for which the iterative procedure converges.

In figure 4 we show a representative sample of drop shapes, for different values of the parameter $\Omega \equiv 4\pi\mu C\surd(S)/T$. For small values of $\Omega$, the drop is shaped like an ellipse. This is predicted by a small perturbation analysis, but the correspondence persists to values of $\Omega$ of order one. However, as the maximum value of $\Omega$ is approached ($\sim 4{\cdot}1$) the central portion of the drop flattens, and beyond the maximum (right-hand branch) a portion of the interface becomes concave. The numerical results suggest that the appearance of a point of vanishing curvature coincides with the maximum, but we have no other evidence for this.

A question we have not explored is the limiting shape corresponding to a drop of vanishing thickness. The reason for this is that, the more slender the drop, the more difficult it becomes to achieve convergence of the numerical iteration. Our primary concern was to generate solutions at least a little way beyond the response curve maximum, rather than explore the problem fully. As a consequence, no major effort was made to overcome these numerical difficulties.

The solutions we have obtained have at least one thing in common with the asymptotic solutions described by Buckmaster (1973) for axisymmetric drops. That is, the response curve has a turning point (maximum). Moreover, the solutions in the neighbourhood of the maximum are connected, in a physically sensible way (the left-hand branch), to the undisturbed state. The implication is, that, as $\Omega$ is slowly increased from zero, there comes a point ($\Omega \sim 4{\cdot}1$) when something dramatic *must* happen. Either the flow must become unsteady, or it must change in a discontinuous fashion. This can not be *solely* a creature of our plane-flow model, since it is not present in Richardson's (1968) solutions for an inviscid plane drop. Mathematically, the presence of a maximum appears to be linked with whether or not the drop is viscous. Since bursting is apparently linked with the *same* fact, we suggest that the maximum coincides with the onset of bursting.

It remains to provide some physical explanation for the bursting phenomenon. To this end, we estimate the forces that are acting on the drop. Such estimates are, in reality, just guesses, but nevertheless they can be illuminating. What we want to do is estimate the forces, in the $x$ direction, acting on one half of the drop (figure 5). These estimates are based, arbitrarily, on the assumption that the pressure just to the left of the point $O$ (inside the drop) is zero. There are several contributions to the net force, but they can be divided into two groups: those that are proportional to $T$, the surface tension, and those that are proportional to $C$, the applied rate of strain. Surface tension tends to preserve the drop, that is, resists the deformation, whereas the straining tends to pull the drop apart. Consequently, the net surface tension force acts to the *left*, whereas the net straining force acts to the *right*, and for equilibrium these two forces must balance.

Suppose, to start off with, that the drop is inviscid. Then the velocity gradient in the outer flow is characterized by $C$, and the normal stress acting on the interface contains terms which are of order $T/b$ and $\mu_0 C$. The corresponding horizontal force has components $O(T)$ and $O(\mu_0 Cb)$. In addition there is a force $2T$ acting to the left, so that equilibrium is apparently a result of a balance of the form

$$T \sim \mu_0 Cb.$$

Such a balance is always possible, no matter how large $C$ may be, since the drop is always free to deform so that $b$ is appropriately small. This must be why inviscid drops do not burst.

The situation is a little more complicated when the drop viscosity is finite. However, we can readily estimate the pressure inside the drop. The interior flow

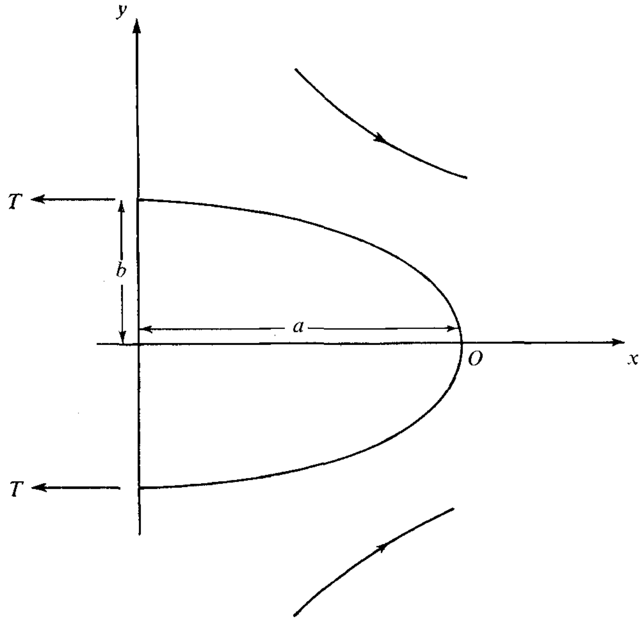


FIGURE 5. Force balance.

consists essentially of four eddies and $u$ is $O(Ca)$ for these eddies. It follows that $\partial^2 u/\partial y^2$ is of order $Ca/b^2$, so that the pressure is $O[\mu_i(Ca^2/b^2)]$ and this corresponds to a force of order $\mu_i(Ca^2/b)$. A similar estimate is valid for the net shear force acting on the interface. This suggests that equilibrium is a result of a balance of the form

$$T \sim \mu_o Cb + \mu_i(Ca^2/b).\dagger$$

True, additional terms may plausibly be advanced for inclusion in the right side of this balance, but the crucial point is that there are disruptive forces that *decrease* with decreasing $b$ and disruptive forces that *increase* with decreasing $b$. Presumably, for moderate deformations, the former forces dominate and an increase in $C$ may be compensated by a decrease in $b$. However, because of the steadily growing importance of the second kind of term as $b$ decreases, there is clearly a limit to this process, and eventually no additional compensation will be possible. Bursting must then occur. Furthermore, since bursting coincides with the emergence, in a dominant role, of disruptive forces that increase as $b$ decreases, it can be anticipated that the manifestation of bursting is a drop of ever-increasing length. This is consistent with experiment and also with the unsteady slender-drop analysis, which predicts, ultimately, exponential growth in the length.

The non-uniqueness revealed by our results is likely to be of mathematical interest only. The reason for this is that, intuitively, we would expect the right-

† Such a balance is also appropriate for axisymmetric drops, and it may be checked with what is known about slender drops. Suppose that $b/a$ is of order $\epsilon$, where $\epsilon \ll 1$; then all three terms contribute to the balance provided that $T/\mu_o Ca$ is $O(\epsilon)$ and $\mu_i/\mu_o$ is $O(\epsilon^2)$. These are precisely the orders of magnitude needed for the slender-drop analysis.

hand solution branch to be unstable. After all, it corresponds to an *increasing* deformation arising out of a decreasing rate of strain. This intuitive feeling is grounded on the following argument. The solution branch lying to the left of the maximum (figure 3) is characterized by a *decrease* in the disruptive force with decreasing values of $b/a$ ($C$ fixed); this is why an *increase* in $C$ corresponds to a decrease in $b/a$ for equilibrium. On the other hand, the right-hand branch is characterized by an *increase* in the disruptive force with decreasing $b/a$ ($C$ fixed); so that in this case a *decrease* in $C$ corresponds to a decrease in $b/a$ for equilibrium. Now stability requires that, for fixed $C$, a deformation of the drop gives rise to a restoring force. But if the deformed drop is thinner than the equilibrium shape, say, then on the right-hand branch there is an *increase* in the disruptive force which will *increase* the deformation. For this reason, the right-hand branch will never be realized in practice.

Thanks are due to John Rinzel for helpful discussions on the numerical aspects of this work, which was performed at the AEC Computing Centre of the Courant Institute of Mathematical Sciences.

## REFERENCES


Brazier-Smith, P. R. 1971 *Phys. Fluids*, **14**, 1.

Buckmaster, J. D. 1972 *J. Fluid Mech.* **55**, 385.

Buckmaster, J. D. 1973 *J. Appl. Mech.* E **40**, 18.

Keller, J. B., Rubenfeld, L. A. & Molyneux, J. E. 1967 *J. Fluid Mech.* **30**, 97.

Mancino, O. G. 1967 *J. A.C.M.* **14**, 341.

Richardson, S. 1968 *J. Fluid Mech.* **33**, 475.

Robinson, S. M. 1966 *SIAM J. Numer. Anal.* **3**, 650.

Rumscheidt, F. D. & Mason, S. G. 1961 *J. Colloid Sci.* **16**, 238.

Taylor, G. I. 1934 *Proc. Roy. Soc.* A **146**, 501.

Taylor, G. I. 1964*a* *Proc. 11th Int. Congr. Appl. Mech., Munich*, p. 790. Springer.

Taylor, G. I. 1964*b* *Proc. Roy. Soc.* A **280**, 383.

Wegener, P. P., Sundell, R. E. & Parlange, J.-Y. 1971 *Z. Flugwiss.* **19**, 347.